21-07-2020

Name of College – S. S. College J. Bad

Depth – Mathematics

Topic – Problem on Pedal Equation (Diff. Calculus) (Tangent and Normal)

Class – B.Sc I (Hons)

Time – 10.15 AM to 11.00 A.M + 11 A.M to 11.45 A.M

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Problem :→ Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with regard to its focus is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

Solution :→ Here Given Equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

⇒ focus of the ellipse is $(ae, 0)$

By Transfering the origin to the focus $(ae, 0)$ the Equation of ellipse becomes

$$\frac{(x+ae)^2}{a^2} + \frac{(y+0)^2}{b^2} = 1$$

$$\Rightarrow \frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——— (1)}$$

the co-ordinate of any point on the ellipse (1) is

$$x + ae = a\cos\theta$$

$$y = b\sin\theta$$

$$\Rightarrow x = a\cos\theta - ae \quad \text{and } y = b\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

Therefore, the equation of tangent at (x, y) is

$$\frac{y - b\sin\theta}{x - (a\cos\theta - ae)} = -\frac{b}{a}\cot\theta \qquad \left[\text{we know } \frac{y-y_1}{x-x_1} = \text{slope}\right]$$

$\Rightarrow ay - ab\sin\theta = -b\cot\theta\, x + (a\cos\theta - ae)\cot\theta$

$\Rightarrow b\cot\theta\, x + ay = ab\sin\theta + b\cot\theta(a\cos\theta - ae)$

$\Rightarrow b\cot\theta\, x + ay - [ab\sin\theta + b\cot\theta(a\cos\theta - ae)] = 0$

Now if p = length of the perpendicular from the origin on the tangent

$$\Rightarrow p = \frac{ab\sin\theta + b\cot\theta(a\cos\theta - ae)}{\sqrt{b^2\cot^2\theta + a^2}}$$

$$\Rightarrow p = \frac{[ab\sin\theta + b\cot\theta(a\cos\theta - ae)]\sin\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\Rightarrow p = \frac{ab\sin^2\theta + b\cos\theta(a\cos\theta - ae)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\Rightarrow p = \frac{ab(\sin^2\theta + \cos^2\theta) - abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\Rightarrow p = \frac{ab(1 - e\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\Rightarrow p^2 = \frac{a^2b^2(1 - e\cos\theta)^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2b^2(1 - e\cos\theta)^2} \quad \text{—— (11)}$$

$r^2 = x^2 + y^2 \qquad \because x = a\cos\theta - ae, \quad y = b\sin\theta$

$= (a\cos\theta - ae)^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2(1 - \cos^2\theta)$

$= (a^2 - b^2)\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2$

$= (a^2-b^2)\cos^2\theta - 2a^2e\cos\theta + (a^2-b^2) + b^2$

$$\Rightarrow r^2 = a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$$
$$= a^2(e^2\cos^2\theta - 2e\cos\theta + 1)$$
$$r^2 = a^2(1 - e\cos\theta)^2$$
$$\therefore r = a(1 - e\cos\theta)$$

Since $\frac{1}{p^2} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab - abe\cos\theta)^2}$

$$= \frac{a^2(1-\cos^2\theta) + b^2\cos^2\theta}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2 - \cos^2\theta(a^2 - b^2)}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2 - a^2e^2\cos^2\theta}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2(1 - e^2\cos^2\theta)}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2(1 + e\cos\theta)(1 - e\cos\theta)}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2\left\{1 + \frac{a-r}{a}\right\}\frac{r}{a}}{(ab - abe\cos\theta)^2}$$
$$= \frac{a^2\left[\frac{a + a - r}{a}\right]\cdot\frac{r}{a}}{(ab - abe\cos\theta)^2}$$
$$= \frac{(2a - r)r}{(ab - abe\cos\theta)^2} \quad \text{using } r = a(1 - e\cos\theta)$$
$$= \frac{(2a - r)r}{a^2b^2[1 - e\cos\theta]^2}$$
$$= \frac{(2a - r)r}{a^2b^2\left[1 - \frac{a-r}{a}\right]^2}$$
$$= \frac{(2a - r)r}{\cancel{a^2}b^2 \cdot \frac{r^2}{\cancel{a^2}}} = \frac{(2a - r)r}{b^2r^2}$$

Thus $\frac{1}{p^2} = \frac{(2a-r)r}{b^2r^2} = \frac{2a-r}{b^2r}$

$\therefore \frac{b^2}{p^2} = \frac{2a-r}{r} = \frac{2a}{r} - 1$

Hence the result

Problem 2:→ In the Equilateral hyperbola $x^2-y^2=a^2$ Prove that $pr=a^2$

Solution:→ Here Equation of the ~~curve~~ equilateral hyperbola

$F(x,y) \equiv x^2-y^2-a^2=0$

⇒ Making the Equation Homogeneous

we have $F \equiv x^2-y^2-a^2z=0$ where $z=1$

$F_x = 2x \quad F_y = -2y$

$F_z = -2a^2z \quad \because z=1$

$= -2a^2$

$\therefore p^2 = \frac{F_z^2}{F_x^2+F_y^2} = \frac{4a^4}{4x^2+4y^2}$

$\Rightarrow p^2 = \frac{a^4}{x^2+y^2} \qquad \because x^2=a^2+y^2 \quad \therefore y^2 = x^2-a^2$

$= \frac{a^4}{x^2+x^2-a^2}$

$= \frac{a^4}{2x^2-a^2}$ ——— (I)

Also $r^2 = x^2+y^2$ ——— (II)

$2x^2-a^2 = r^2$

$\therefore p^2 = \frac{a^4}{r^2} \Rightarrow p = \frac{a^2}{r}$

$\Rightarrow pr = a^2$

This is the required pedal Equation.

Show that the Pedal Equation of the Astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is $r^2=a^2-3p^2$

or Find the Pedal Equation of Curve

$x=a\cos^3\theta$, $y=a\sin^3\theta$

$\Rightarrow \frac{x}{a}=\cos^3\theta$ $\qquad y/a=\sin^3\theta$

$\Rightarrow \left(\frac{x}{a}\right)^{1/3}=\cos\theta$ $\qquad (y/a)^{1/3}=\sin\theta$

$\Rightarrow \left(\frac{x}{a}\right)^{2/3}=\cos^2\theta$ $\Rightarrow (y/a)^{2/3}=\sin^2\theta$

$\Rightarrow \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=\sin^2\theta+\cos^2\theta=1$

$x^{2/3}+y^{2/3}=a^{2/3}$

Solution $\Rightarrow$ Now By the question, Equation of astroid

$F(x,y)\equiv x^{2/3}+y^{2/3}-a^{2/3}=0$

$f_x=\frac{2}{3}x^{-1/3}$ $\qquad f_y=2/3\,y^{-1/3}$

$$p^2=\frac{[xf_x+yf_y]^2}{f_x^2+f_y^2}$$

$$=\frac{\left[x\left(\frac{2}{3}x^{-1/3}\right)+y\left(\frac{2}{3}y^{-1/3}\right)\right]^2}{\left(\frac{2}{3}x^{-1/3}\right)^2+\left(\frac{2}{3}y^{-1/3}\right)^2}$$

$$=\frac{\left[\frac{2}{3}x^{2/3}+\frac{2}{3}y^{2/3}\right]^2}{\frac{4}{9}x^{-2/3}+\frac{4}{9}y^{-2/3}}$$

$$=\frac{\cancel{\frac{4}{9}}\left[x^{2/3}+y^{2/3}\right]^2}{\cancel{\frac{4}{9}}\left[\frac{1}{x^{2/3}}+\frac{1}{y^{2/3}}\right]}=\frac{(a^{2/3})^2}{a^{2/3}}x^{2/3}\cdot y^{2/3}$$

$$=a^{2/3}x^{2/3}y^{2/3}$$

$p^2=a^{2/3}x^{2/3}y^{2/3}$

$\Rightarrow 3p^2=3a^{2/3}x^{2/3}y^{2/3}$

Now $a^2 - r^2 = a^2 - (x^2 + y^2)$

$$= a^2 - \left[(x^{2/3} + y^{2/3})^3 - 3x^{2/3} \cdot y^{2/3}(x^{2/3} + y^{2/3})\right]$$

$$= a^2 - \left[(a^{2/3})^3 - 3x^{2/3} y^{2/3} \cdot a^{2/3}\right]$$

$$= \cancel{a^2} - \cancel{a^2} + 3x^{2/3} \cdot y^{2/3} \cdot a^{2/3}$$

$$\Rightarrow a^2 - r^2 = 3x^{2/3} \cdot y^{2/3} a^{2/3}$$

$$= 3p^2 \qquad \because 3x^{2/3} y^{2/3} a^{2/3} = 3p^2$$

$\therefore r^2 + 3p^2 = a^2$. this is the required Pedal Equation